

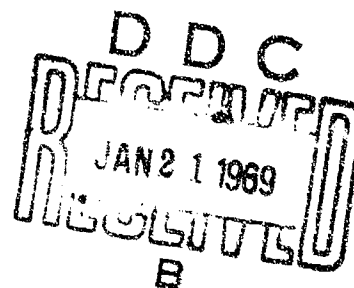
AD 680599

RESEARCH PAPER P-413

## DISCRETE RENEWAL PROCESSES

Michael Muntner

December 1968



INSTITUTE FOR DEFENSE ANALYSES  
SCIENCE AND TECHNOLOGY DIVISION

IDA Log No. HQ 68-8701  
Copy **059** of 100 Copies

This document has been approved  
for public release and sale; its  
distribution is unlimited

RESEARCH PAPER P-413

## DISCRETE RENEWAL PROCESSES

Michael Muntner

December 1968

This document has been approved for public release and sale;  
its distribution is unlimited.



INSTITUTE FOR DEFENSE ANALYSES  
SCIENCE AND TECHNOLOGY DIVISION  
400 Army-Navy Drive, Arlington, Virginia 22202

Contract DAHCl5 67 C 0011  
Task T-10

# ABSTRACT

Discrete renewal processes, until recently, have not been applied to the mathematical modelling of physical processes. Analyses of such renewal processes have proceeded on the basis of generating functions but the results are often too complicated to be of use. This paper presents an alternative approach to discrete renewal theory and calculates many of the more complex statistics of such processes.

## CONTENTS

I. Introduction	1
II. Elementary Statistics	3
III. Complex Statistics	8
A. Compound Counting Distribution	8
B. Aging	10
C. Bursts of Failures	11
D. Interleaving	12
E. Freeze-Out Problems	13
References	20

## I. INTRODUCTION

A renewal process is one in which events occur at time  $t_j$  according to the behavior of some underlying stochastic mechanism. When the event occurs, the stochastic mechanism is renewed (i.e., replaced by a new mechanism identical to, but statistically independent of, the previous ones.)

The mathematical analysis of reliability and queuing problems has resulted in a great deal of literature devoted to continuous time renewal theory (i.e., the events can occur at any time). Aside from the volumes devoted to the applications of the theory, virtually all books on stochastic processes devote at least a section, if not a chapter, to its discussion.

Discrete time renewal processes (i.e., the events can occur at only fixed times) have, unfortunately, not enjoyed such a wide exposure either by virtue of their apparent intractability or the lack of an obvious application. Recently, however, applications<sup>1,2</sup> have arisen wherein a discrete time renewal process appears to be quite useful. To be specific, the error process in digital communications systems has been modelled by a discrete time renewal process. Another process being modelled by a renewal process, at Bell Telephone Laboratories, is the typing of characters on teletypewriters. This latter application is quite important insofar as efficient communications systems are being sought for the remote programming of time-shared computers.\*

---

\*While present typewriters are basically asynchronous, new Bell System models will operate in a synchronous manner.

The discrete time renewal process would, no doubt, be quite useful in the modelling and analysis of those systems in which events were fixed to occur at specific instants in time. The growth of digital technology has fostered the creation of such systems in communications and automata. It is the purpose of this paper\*, therefore, to investigate the statistics of such processes in order to facilitate their use in practical problems.

The statistical mechanism governing the renewal events can be described in the following manner. Consider the non-negative random variable  $X$ , which for the purposes of discussion, is called the failure time of a component. This variable is the length of time between renewal events. The distinction between the continuous time and the discrete time theory is made as follows:

- (a) The random variable has a continuous distribution over the range  $(0, \infty)$ , its distribution being determined by a probability density function,  $f(x)$ . This is the continuous case and is discussed in great detail by Cox<sup>3</sup> and will not be examined here.
- (b) There is a constant,  $T$ , such that the only possible values of  $X$  are  $(T, 2T, \dots)$ . The process is determined by its gap length distribution,  $p(j)$ , which is the probability that  $X = jT$ . This latter case is the discrete renewal process. Feller<sup>4,5</sup> devotes a far from insignificant portion of his books to this theory, but his work does not lend itself to practical (e.g., actually calculating numbers) applications.

The approach taken by Feller, as well as Haight<sup>6</sup>, has been to use the generating function of  $X$ . This paper will present an alternative way of looking at discrete time renewal processes and, in particular, will present relatively simple expressions for some of the more complicated statistics of such processes.

---

\*The work on this paper was done in the period March-June 1968.

## II. ELEMENTARY STATISTICS

The discrete renewal process, by definition, has gaps (length of time between failures) whose lengths are independent and are distributed according to a common distribution. Let  $p(j)$  be the probability that following a failure, the next failure occurs at  $jT$ . For ease of description, a trial is said to be made at every  $T$ . That is,  $p(j) = P(0^{j-1} 1|1)$  where 1 denotes a failure, 0 denotes a non-failure\* and  $0^i$  corresponds to  $i$  successive zeroes. The gap length distribution has the following properties:

$$\sum_{j=1}^{\infty} p(j) = 1 \quad (1)$$

and that the average failure rate,  $p_1$ , is equal to the reciprocal of the average distance between failures.

$$p_1 = \left[ \sum_{j=1}^{\infty} jp(j) \right]^{-1} \quad (2)$$

An alternative definition of failure rate is

$$p_1 = \lim_{J \rightarrow \infty} E \left\{ \frac{\text{number of failures in } (0, JT)}{J} \right\} .$$

---

\*A non-failure will also be called a success.

Let

$$Q(m) = \sum_{j=m+1}^{\infty} p(j) \quad (3)$$

That is,  $Q(m)$  is the probability of there being at least  $m$  consecutive successes following a failure. Now, consider the event where there are  $m$  successes before a failure (i.e.,  $0^m 1$ ). The probability of this event is

$$P(0^m 1) = p_1 - p_1 \sum_{j=1}^m p(j) = p_1 Q(m) \quad (4)$$

Note that  $P(0^m 1) = P(1 0^m)$ . This facilitates the calculation of the probability of some rather complicated events.

In many applications, the probability of  $m$  failures out of  $n$  trials,  $P(m, n)$ , is important.\* This is often called the counting statistic. Elliott<sup>7</sup> has developed an equation for  $P(m, n)$  for a renewal process.

$$P(m, n) = \sum_{j=1}^{n-m+1} p_1 Q(j-1) R(m, n-j+1) \quad 1 \leq m \leq n \quad (5)$$

where  $R(m, n)$  is the probability that  $m-1$  failures occur in the  $n-1$  trials following a failure. Therefore,  $R(1, n) = Q(n-1)$  for  $n \geq 1$  and

$$R(m, n) = \sum_{j=1}^{n-m+1} p(j) R(m-1, n-j) \quad 2 \leq m \leq n \quad (6)$$

---

\*This probability assumes no knowledge of the failures in the trials preceding the  $n$  trial sequence under investigation.

It is to be noted that  $P(m,n)$  can be expressed in terms of  $p_1$  and  $p(j)$ , for  $j \leq n$ . Several of the above parameters and distributions are analogous to those in continuous time renewal processes. For example,  $p(j) \approx f(x)$  and  $p_1 \approx \mu$ .

It is at this point that it is worthwhile to compare the simplicity of the above expressions with those available through the use of generating functions. Let  $g(z)$  and  $G(z)$  be the generating functions associated with  $p(j)$  and  $Q(j)$ , respectively. That is,

$$g(z) = \sum_{j=1}^{\infty} p(j) z^j \quad (1)$$

and

$$G(z) = \sum_{j=1}^{\infty} Q(j) z^j \quad (2)$$

From Equation 3

$$G(z) = \frac{1-g(z)}{1-z} \quad (3)$$

Letting

$$H_m(z) = \sum_{n=m}^{\infty} P(m,n) z^n \quad (4)$$

it follows

$$H_m(z) = \frac{1}{1-z} (z) [g(z)]^{m-1} (z) \quad (5)$$

or alternatively

$$H_m(z) = p_1 z \left[ \frac{1-g(z)}{1-z} \right]^m [g(z)]^{m-1} . \quad (12)$$

Calculation of  $P(m,n)$ , as Elliott puts it, is rather inconvenient when compared to Equation 5. Generating functions do turn out to be quite useful for certain problems in renewal theory, but it is noted that the recurrence equation approach for  $P(m,n)$  is ideally suited for calculations done on a computer.

In addition to the gap length distribution, the "autocorrelation" of the failures is often quite useful. The autocorrelation,  $a(j)$ , is defined as the probability that, following a failure, a failure occurs at  $X = jT$ . For a renewal process, however, the autocorrelation can be derived from the gap length distribution.

$$a(j) = p(j) + \sum_{s=1}^{j-1} p(s) a(j-s) \quad \text{for } j > 1 \quad (13)$$

where  $a(1) = p(1)$  and  $a(0) \equiv 1$ . Therefore a renewal process can be described by  $p(j)$ ,  $a(j)$  or  $P(m,n)$ . Feller<sup>4</sup> has also proven that

$$\lim_{j \rightarrow \infty} a(j) = p_1.$$

A renewal process of interest is one in which

$$a(j) = \alpha \beta^j + p_1 \quad j \geq 1 \quad \beta < 1 . \quad (14)$$

This is the autocorrelation Gilbert<sup>8</sup> obtains in his Markov model. It happens that this simple model generates a renewal process while the

Berkovits<sup>9</sup> model, which has the same autocorrelation, is not a renewal process.\* For a renewal process with this autocorrelation, and if  $p_1 \ll \alpha \beta^j$  for  $j \leq N$ , then

$$a(j) \approx \alpha \beta^j . \quad (15)$$

From Equation 13

$$p(j) = \alpha \beta^j (1-\alpha)^{j-1} . \quad (16)$$

Then if  $\frac{\alpha \beta}{1-\beta(1-\alpha)} \approx 1$ ,

$$Q(m) = \frac{\alpha \beta^{m+1} (1-\alpha)^m}{[1-\beta(1-\alpha)]} \quad (17)$$

and

$$R(m,n) = \binom{n-1}{m-1} \frac{\alpha^m \beta^n (1-\alpha)^{n-m}}{[1-\beta(1-\alpha)]} . \quad (18)$$

Using Equation 5

$$P(m,n) = p_1 \binom{n}{m} \frac{\alpha^{m+1} \beta^{n+1} (1-\alpha)^{n-m}}{[1-\beta(1-\alpha)]^2} \quad 1 \leq m \leq n . \quad (19)$$

At this point, it is apparent that a Bernoulli process (i.e., one in which each trial is independent of the others) is a renewal process with

$$a(j) = p ,$$

$$p(j) = p(1-p)^{j-1} , \quad (20)$$

and

$$P(m,n) = \binom{n}{m} p^m (1-p)^{n-m} .$$

---

\*The fact that processes have the same autocorrelations does not imply that they have the same gap length distributions unless they are both renewal processes.

### III. COMPLEX STATISTICS

#### A. COMPOUND COUNTING DISTRIBUTION

The fact that there may be a correlation between failures raises the importance of the compound counting statistic,  $P(m_1, m_2, \dots, m_N, N)$ . This probability corresponds to the following: Consider a sequence of  $nN$  trials, as shown in Fig. 1, which are divided into  $N$  subsequences of  $n$  trials each. In the  $i^{\text{th}}$  subsequence there are  $m_i$  failures

$$0 < m_i \leq n.$$

The first failure in the  $i^{\text{th}}$  subsequence is preceded by  $\ell_i - 1$  successes and the last failure is followed by  $a_i - \ell_i$  successes. The failures are clustered in an interval of length  $n - a_i + 1$ . This continues until the last  $n$  trials where  $m_N = 1$  failures occur in the last  $m - \ell_N$  trials though not necessarily ending in a failure. The black bars show the beginning and end of an  $m_i$  failure cluster in each  $n$  trial subsequence. To examine this case, first let  $S(m, n)$  be the probability that  $(m - 1)$  failures occur in the  $(n - 1)$  trials following a failure, and that the  $(m - 1)^{\text{st}}$  failure is at the  $(n - 1)^{\text{st}}$  trial.  $S(m, n)$  obeys the following recurrence relation

$$S(m, n) = \sum_{j=1}^{n-m+1} S(m-1, n-j) p(j) \quad (1)$$

for  $2 \leq m \leq n$  where  $S(1, 1) = 0$  and  $S(1, n) = 0$  for  $n > 1$ .  $R(m, n)$  introduced in Equation 5 follows directly from  $S(m, n)$  since

$$R(m,n) = \sum_{j=1}^{n-m+1} \sum_{i=j}^{\infty} S(m,n-j+1) p(i) \quad (22)$$

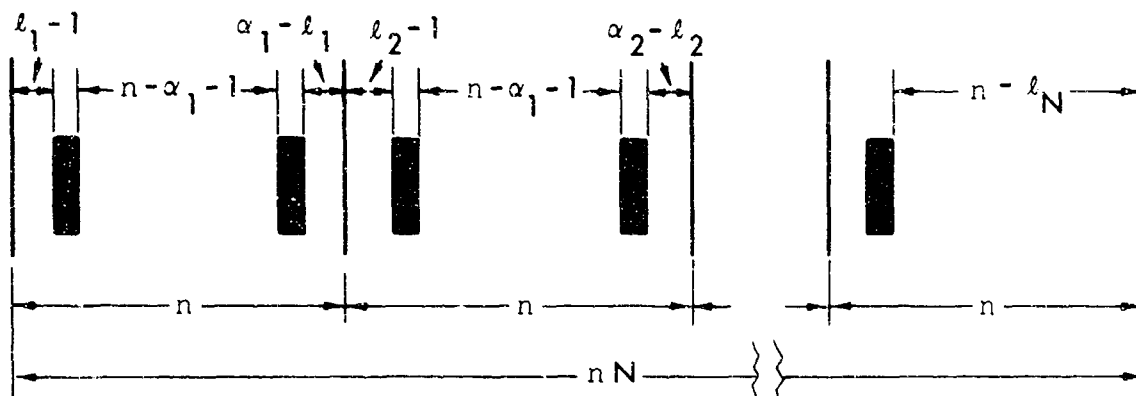


FIGURE 1. Failure Sequence for Complex Counting Statistics

The general expression for the compound counting statistic is

$$P(m_1, m_2, \dots, m_N, N) = p_1 \sum_{2N-1} \dots \sum_{2N-1} \quad (23)$$

$$Q(l_1-1) \left[ \prod_{j=1}^{N-1} S(m_j, n-a_j+1) p(a_j-l_j+l_{j+1}+1) \right] R(m_N, n-l_N+1)$$

where  $m_i > 0$ . The first  $(N-1)$  pairs of summations over  $a_j$  and  $l_j$  are summed as follows:

$$\sum_{l_j=1}^{n-m_j+1} \sum_{a_j=l_j}^{n-m_j+1} \quad \text{for } 1 \leq j \leq N-1$$

and the last summation is

$$\sum_{l_N=1}^{n-m_N+1}$$

Suppose  $m_j = 0$  (i.e., a group of trials has no failures). This can happen in three ways:

- (i) If the  $n$  trial block is the first block, omit  $S(m_1, n-a_1+1)$  and  $p(a_1-l_1+l_2+1)$  and the associated summations and change  $Q(l_1-1)$  to  $Q(l_1+n-1)$ .
- (ii) If the  $n$  trial block is the last block, then omit the term  $R(m_N, n-l_N+1)$  and sum the last term  $p(a_{N-1}-l_{N-1}+l_N+1)$  over the limits  $m \leq l_N < \infty$ .
- (iii) If the  $n$  trial block is internal, omit the respective  $S(m_j, n-a_j+1)$   $p(a_j-l_j+l_{j+1}+1)$  term as well as the corresponding summations over  $a_j$  and  $l_{j+1}$  and change  $p(a_{j-1}-l_{j-1}+l_j+1)$  to  $p(a_{j-1}-l_{j-1}+l_j+n+1)$ . The above process of removing terms can be continued up to the point where only one  $m_j \neq 0$  because Equation 23 assumes at least one renewal event. For all  $m_j = 0$ ,

$$P(0, \dots, 0, N) = p_1 \sum_{j=mN}^{\infty} Q(j) .$$

## B. AGING

An interesting statistic often calculated for continuous renewal processes is the age-specific failure rate. This is defined as the probability of a failure between  $x$  and  $x+\Delta x$  given it has not failed up to  $x$ . For the discrete case it is defined as the probability of a failure on the  $j^{\text{th}}$  trial given no failures up to, and including, the  $(j-1)^{\text{st}}$  trial with no knowledge of the trials preceding these  $j$  trials. Denoting it by  $\varphi(j)$ , one gets

$$\varphi(j) = \frac{Q(j-1)}{\sum_{m=j-1}^{\infty} Q(m)} . \quad (24)$$

If  $\phi(j)$  increases with  $j$  it is said to have positive aging and becomes more likely to fail. Some processes have a  $\phi(j)$  that decreases with  $j$ . There is however a process with no aging. That is, let

$$\phi(j) = \lambda$$

or

(25)

$$Q(j-1) = \lambda \sum_{m=j-1}^{\infty} Q(m) .$$

Noting that  $Q(0) = 1$  and  $\sum_{m=0}^{\infty} Q(m) = \frac{1}{p_1}$  yields

$$\lambda = p_1$$

and

(26)

$$p(j) = p_1 (1-p_1)^{j-1} ,$$

which is obviously the Bernoulli process.

### C. BURSTS OF FAILURES

A simple extension of the previous discussion is to consider bursts of failures. A burst of length  $m$  is defined to be  $m$  consecutive trials beginning and ending in a failure. This does not require all  $m$  trials to be failures, but only the first and last. When considering  $n$  consecutive trials, the probability of it containing a burst of length  $m$  is denoted by  $B(m,n)$ . Bursts of failures are of considerable interest in many physical processes which may be modelled. For example, errors in digital communications are noted to be clustered (i.e., come in bursts) and error correcting codes are constructed to deal with these bursts. It is also noted that the hitting of typewriter keys, which in this case corresponds to failure, occurs in bursts. The examination of bursts of failures is therefore important.

For the process described previously

$$\begin{aligned}
 B(m,n) = & p_1[a(m-1)] [p(n-m+1) + p(n-m+2)+...] \\
 & + [1-p(1)] p_1 [a(m-1)] [p(n-m)+p(n-m+1)+...] \quad (27) \\
 & + ...+[1-p(1)-p(2)-...-p(n-m)] p_1 [a(m-1)]
 \end{aligned}$$

The first term in the above expression corresponds to the first failure of the burst in the first of the  $n$  trials; the second term to where the first failure is at the second trial; and the last term is where the first failure of the burst is on the  $(n-m+1)^{st}$  trial. This can be compactly written as

$$B(m,n) = p_1 a(m-1) \sum_{j=0}^{n-m} Q(j)Q(n-m-j) \quad (28)$$

#### D. INTERLEAVING

A technique often used to overcome the effects of bursts is to interleave the process. For example, in the case of digital transmission, time division multiplexing  $M$  data streams such that two digits originally adjacent in any one stream are transmitted  $M$  digits apart spreads a burst of errors over the  $M$  streams and makes the errors look almost as if they were randomly distributed in any one data stream. For the remote programming application,  $M$  parallel computer input ports would be able to digest a burst of teletypewriter characters if each port accepted every  $M^{th}$  character. The problem is now to analyze the process obtained by examining every  $M^{th}$  trial of a discrete renewal process.

The autocorrelation of the events in the interleaved process,  $a(j,M)$ , is obviously  $a(jM)$ . The gap length distribution is now given by

$$p(j,M) = a(jM) - \sum_{s=1}^{j-1} a(sM) p(j-s,M) \quad j > 1 \quad (29)$$

As M gets large,

$$\lim_{M \rightarrow \infty} a(jM) = p_1 \quad (30)$$

Therefore

$$\lim_{M \rightarrow \infty} p(j,M) = p_1(1-p_1)^{j-1} \quad (31)$$

and the process becomes a Bernoulli process.

#### E. FREEZE-OUT PROBLEMS

The freeze-out problem arises when a physical device requires a specific time interval to digest recorded data and cannot accept additional data during that time interval. For example, an event occurs randomly in time. When it occurs a "counter" must record some appropriate data. This recording period lasts, say, n seconds (or trials) and if another event occurs during these n seconds it goes unrecorded. The probability of such an event for a discrete renewal process being unrecorded is not examined directly but rather two different measures are used: the mean time to an unrecorded event and the probability of at least one unrecorded event in N trials.

##### 1. Mean Time to Unrecorded Event

If the probability that the first unrecorded event occurs on the  $j^{\text{th}}$  trial is denoted by  $P_j$ , the mean time to failure,  $\bar{T}$ , is given by

$$\bar{T} = \sum_{j=1}^{\infty} j P_j \quad (32)$$

The guard space time to prevent a freeze-out is  $n-1$  trials such that if two events occur within any  $n$  trials, then the second is not recorded. In the analysis that follows, the signal flow graph techniques of Sittler<sup>10</sup> and Huggins<sup>11</sup> will be used. The state diagram of Fig. 2 is obtained where:

$\bar{i}$  is the state of having at least  $i$  successes (in this case trials in which events do not occur) before the first failure (event);

$\underline{i}$  is the state of having at least  $i$  successes since the last failure; and

$E_1$  is the state of having a failure but not an unrecorded event.

Then letting  $q(a,b)$  be the probability of going to state  $b$  from state  $a$ ,

$$q(\bar{i}, \bar{i}+1) = \frac{1-p_1 \sum_{m=0}^i Q(m)}{1-p_1 \sum_{m=0}^{i-1} Q(m)} \quad (33)$$

$$q(\bar{i}, E_1) = \frac{p_1 Q(i)}{1-p_1 \sum_{m=0}^{i-1} Q(m)} \quad (34)$$

$$q(\underline{i}, \underline{i}+1) = \frac{Q(i+1)}{Q(i)} \quad (35)$$

$$q(\underline{i}, E_1) = \frac{p(i-1)}{Q(i)} \quad \text{for } i \geq n-1 \quad (36)$$

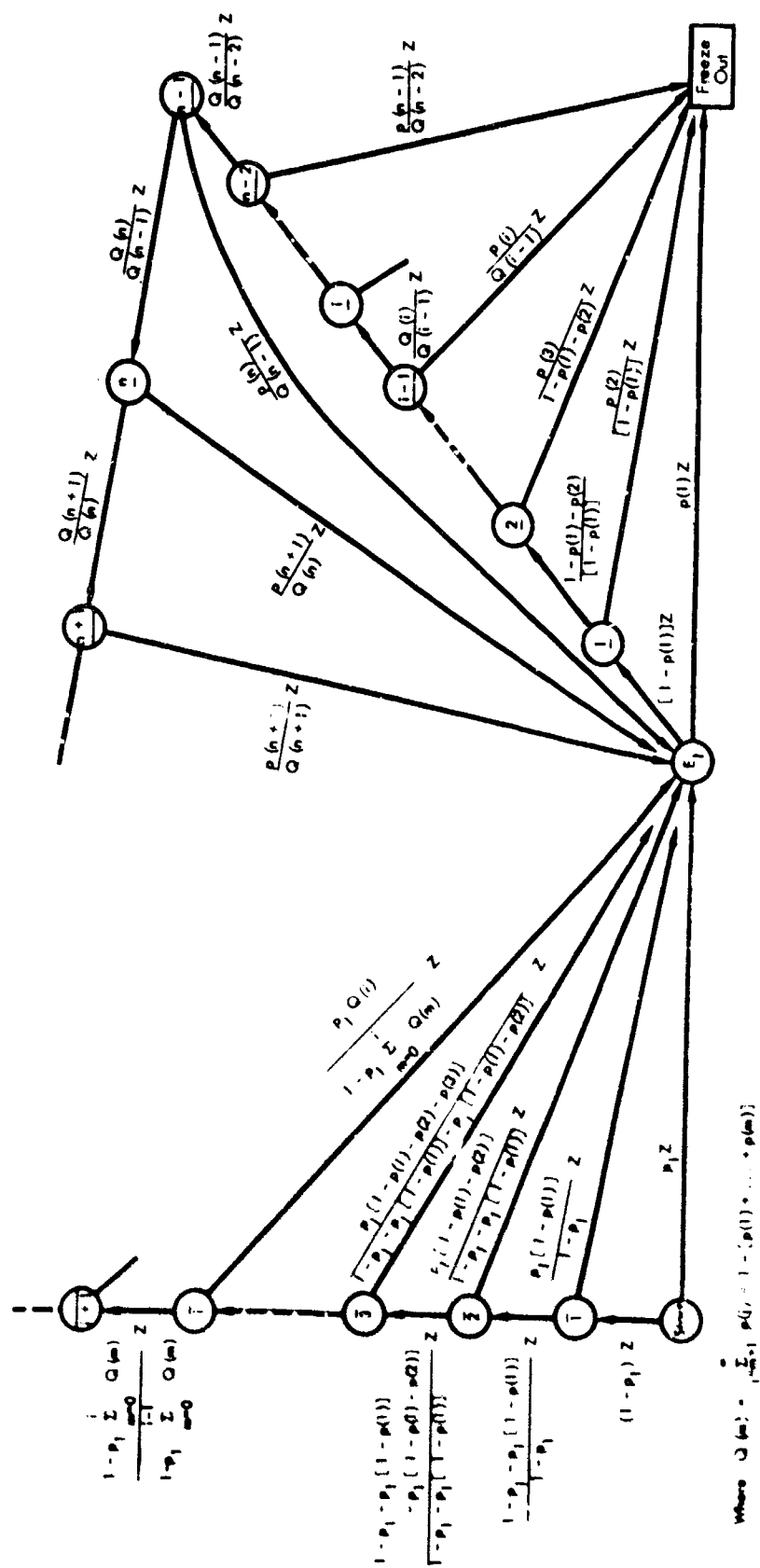


FIGURE 2. State Flow Graph for a Freeze Out of a Counter Requiring a Guard Space of  $n-1$

$$q(\underline{i}, \text{unrecorded event}) = \frac{p(i+1)}{Q(i)} \quad \text{for } i \leq n-2 \quad (37)$$

The flow graph can be reduced to that of Fig. 3. The "transmission from start to freeze-out" is  $P(Z)$ .

$$P(Z) = \frac{p_1 Z \left[ \sum_{m=0}^{\infty} Q(m) Z^m \right] \left[ \sum_{j=1}^{n-1} p(j) Z^j \right]}{1 - \sum_{j=n}^{\infty} p(j) Z^j} \quad (38)$$

If  $P(Z)$  were expanded as

$$P(Z) = P_0 + P_1 Z + P_2 Z^2 + \dots + P_j Z^j + \dots \quad (39)$$

the coefficient of  $Z^j$  is the probability of an unrecorded event on the  $j^{\text{th}}$  trial.

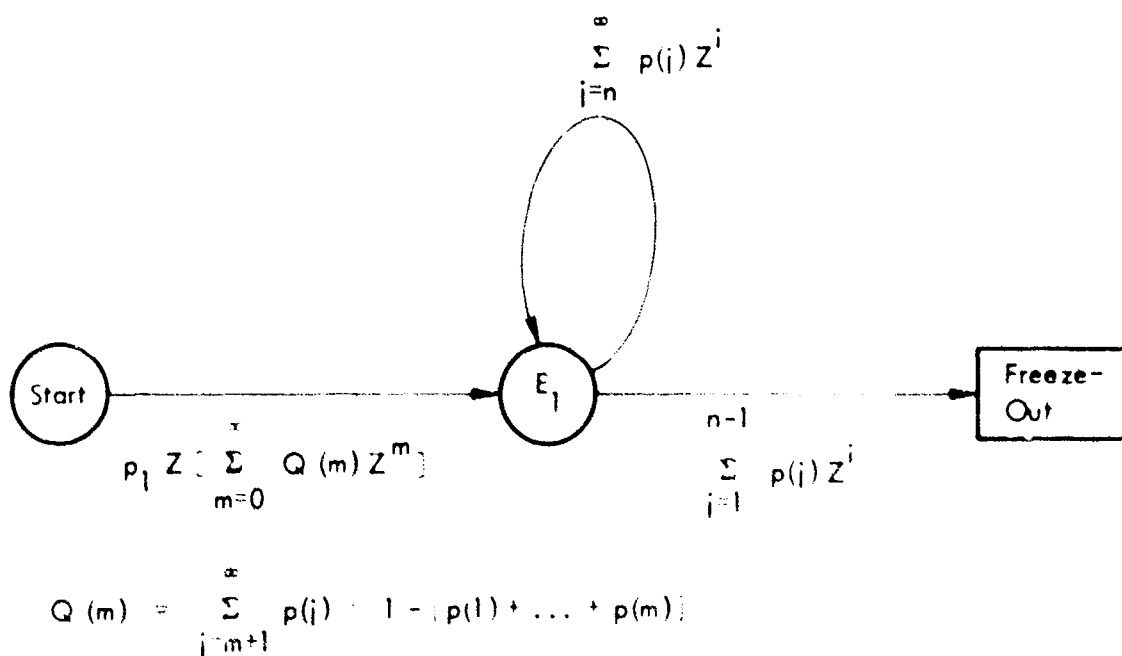


FIGURE 3. Reduced State Flow Graph for a Freeze-Out of a Counter Requiring a Guard Space of  $n-1$

From the definition of  $\bar{T}$ ,

$$\bar{T} = \left. \frac{dP(Z)}{dZ} \right|_{Z=1} \quad (40)$$

and therefore

$$\bar{T} = 1 + \frac{1}{p_1} \frac{1}{[1-Q(n-1)]} + p_1 \sum_{m=0}^{\infty} m Q(m) \quad (41)$$

## 2. Freeze-Out in N Trials

The preceding analysis evaluated the mean time to freeze-out, but this measure may have little significance to someone designing an experiment to record data. As an alternative,  $P(N)$ , the probability of at least one freeze-out in  $N$  trials will be examined.

$$P(N) = \sum_{j=1}^N P_j \quad (42)$$

where

$$P_j = \frac{1}{j!} \left. \frac{d^j P(Z)}{dZ^j} \right|_{Z=0} \quad (43)$$

The complicated nature of  $P(Z)$  precludes this approach.

Another technique is to calculate  $P_j$  directly. For  $j \leq n + 1$ , the only way to have a freeze-out on the  $j^{\text{th}}$  trial is to have only one failure in the first  $j-1$  trials. Then

$$P_j = \sum_{k=0}^{j-2} Q(k) p_1 p(j-k-1) \quad (44)$$

where the above expression has  $k$  successful trials before the first failure. If  $n + 1 < j \leq 2n + 1$ , a freeze-out on the  $j^{\text{th}}$  trial can occur because of a failure in trials  $j - n + 1$  to  $j - 1$ , or by two previous failures. The probabilities are

$$\sum_{k=j-n}^{j-2} Q(k) p_1 p(j-k-1) \quad (45)$$

and  $\sum_{k=0}^{j-n-2} \sum_{r=n}^{j-k-2} Q(k) p_1 p(r) p(j-k-r-1)$ , respectively.

Continuing in this vein becomes extremely difficult when one realizes that if  $in + 1 < j \leq (i + 1)n + 1$  then it is possible to have  $m$  failures prior to the failure causing the freeze-out where  $m = 1, \dots, i + 1$ .

An alternative approach is to work backwards. It was shown in Equation 4 that reversing a pattern of failure does not alter the probabilities involved. Then for a freeze-out on the  $j^{\text{th}}$  trial, there can be at most  $n - 2$  successes preceding it after the previous failure. Preceding that failure there is at least  $n - 1$  successes and so on. The probability of a freeze-out on the  $j^{\text{th}}$  trial is now compactly expressed as

$$P_j = \sum_{r_0=1}^{n-1} \sum_{r_1=n}^{j-(m-1)n} \sum_{r_2=n}^{j-(m-2)n-r_1} \dots \sum_{r_{m-1}=n}^{j-n-r_1-\dots-r_{m-2}} p_1 p(r_0) p(r_1) \dots p(r_{m-1}) Q(j-r_0-r_1-\dots-r_{m-1}-1) \quad (46)$$

where  $m$  is defined above. The only exception to the above occurs when  $m = i + 1$  and  $j = in + a$  where  $1 < a < n$  in which case Equation 46 becomes

$$P_j = \sum_{r_0=1}^{a-1} p_1 [p(n)]^i p(\lambda) Q(j-r_0-1-in) \quad (47)$$

Here again, as in the previous statistics, a complicated probability is obtained in a form directly amenable to computer calculation wherein the only data needed is  $p_1$  and  $p(j)$  for  $j \leq N - 2$ .

## REFERENCES

1. S.M. Sussman, "Analysis of the Pareto Model for Error Statistics on Telephone Circuits," IEEE Transactions on Communications Systems, June 1963.
2. M. Muntner, and J.K. Wolf, "Predicted Performance of Error Control Techniques Over Real Channels," to be published in October 1968 FGIT.
3. D.R. Cox, "Renewal Theory," Metheun's Monograph's on Applied Probability and Statistics, 1962.
4. W. Feller, An Introduction to Probability Theory and Its Applications, John Wiley & Sons, Vol. I, 1961.
5. W. Feller, An Introduction to Probability Theory and Its Applications, John Wiley & Sons, Vol. II, 1966.
6. F.A. Haight, "Counting Distributions for Renewal Processes," Biometrika, Vol. 52, pp 395, 1965.
7. E.O. Elliott, "A Model of the Switched Telephone Network for Data Communications," Bell System Technical Journal, Vol. 44, No. 1, January 1965.
8. E.N. Gilbert, "Capacity of a Burst Noise Channel," Bell System Technical Journal, Vol. 39, No. 5, September 1960.
9. S. Berkovits, E.L. Cohen, and N. Zierler, "A Model for Digital Error Distributions," MITRE Corporation, Project 7560 Report No. TM 4189, 1966.
10. R.W. Sittler, "System Analysis of Discrete Markov Processes," IRE Transactions, Vol. CT-3, 1956.
11. W.H. Huggins, "Signal-Flow Graphs and Random Signals," Proceedings of the IRE, Vol. 45, No. 1, January 1957.

**UNCLASSIFIED**

Security Classification

DOCUMENT CONTROL DATA - R & D		
<small>(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)</small>		
1. ORIGINATING ACTIVITY <small>(Corporate author)</small>		2a. REPORT SECURITY CLASSIFICATION
Institute for Defense Analyses		Unclassified
		2b. GROUP
		--
3. REPORT TITLE		
Discrete Renewal Processes		
4. DESCRIPTIVE NOTES <small>(Type of report and inclusive dates)</small>		
Research Paper P-413 - December 1968		
5. AUTHOR(S) <small>(First name, middle initial, last name)</small>		
Michael Muntner		
6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF PAGES
December 1968	20	11
8a. CONTRACT OR GRANT NO.	9a. ORIGINATOR'S REPORT NUMBER(S)	
DAHCL5 67 C 0011	P-413	
b. PROJECT NO.		
Task T-10		
c.	9b. OTHER REPORT NO(S) <small>(Any other numbers that may be assigned this report)</small>	
d.	NA	
10. DISTRIBUTION STATEMENT		
This document has been approved for public release and sale; its distribution is unlimited.		
11. SUPPLEMENTARY NOTES		12. SPONSORING MILITARY ACTIVITY
NA		NA
13. ABSTRACT		
<p>Discrete renewal process, until recently, have not been applied to the mathematical modelling of physical processes. Analyses of such renewal processes have proceeded on the basis of generating functions but the results are often too complicated to be of use. This paper presents an alternative approach to discrete renewal theory and calculates many of the more complex statistics of such processes.</p>		

DD FORM 1 NOV 68 1473

**UNCLASSIFIED**  
Security Classification